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On 3-colorable planar graphs without prescribed cycles☆

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Abstract

In this paper we prove that every planar graph without cycles of length 4, 5, 6 and 8 is 3-colorable.

Keywords: Coloring; Planar graph; Cycle

1. Introduction

Grötzsch [4] proved that planar graphs without 3-cycles are 3-colorable. Steinberg conjectured that every planar graph without 4-cycles and 5-cycles is 3-colorable [5]. Relaxation of this conjecture, in [7], Erdős asked if there exists an integer k^* such that every planar graph without cycles of length from 4 to k^* is 3-colorable? Abbott and Zhou [1] answered positively the Erdős' question by showing that such k^* does exist and is at most 11. The result has been gradually improved to $k^* \leqslant 9$ by Borodin [2] and, independently, Sanders and Zhao [6], and to $k^* \leqslant 7$ by Borodin et al. [3]. More recently, Xu [8] showed that every planar graph without 5-cycles, 7-cycles and adjacent 3-cycles is 3-colorable, which implies that planar graphs without 4, 5 and 7-cycles are 3-colorable. Moreover, Zhang and Wu [9] even proved 3-choosability of planar graphs without 4, 5, 6 and 9-cycles.

In this paper, we prove the following result:

Theorem 1. *Every planar graph without* 4, 5, 6 *and* 8*-cycles is* 3*-colorable*.

Combining Theorem 1 and the results of [3,9], we obtain:

Corollary 2. *Let G be a planar graph without* 4, 5 *and* 6*-cycles. If G further contains no k-cycles for some fixed* $k \in \{7, 8, 9\}$, *then G is* 3*-colorable.*

Let $\mathscr{G}$ denote the class of plane graphs without cycles of length 4, 5, 6 and 8. Instead of showing Theorem 1, we prove the following stronger result:

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Theorem 3. *Every proper* 3*-coloring of the vertices of any face of degree* 7, 9, 10, 11 *or* 12 *in a connected plane graph in* $\mathscr{G}$ *can be extended to a proper* 3*-coloring of the whole graph.*

We remark that Theorem 3 together with its proof in next section is intensively stimulated by the result of [3].
Assume that Theorem 3 is true. We can give an easy proof for Theorem 1.

Proof of Theorem 1. Suppose that G is a planar graph without 4, 5, 6 and 8-cycles. If G does not contain 7-cycles, then G is 3-colorable by the result of [3]. So suppose that G contains a 7-cycle C. Since G has no cycles of length 4 to 6, C has no chord. Thus, C has a proper 3-coloring c. By Theorem 3, c can be extended both inside and outside of C to produce a proper 3-coloring of G. □

Only simple graphs are considered in this paper. A *plane* graph is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set, order and maximum degree by $V(G)$, $E(G)$, $F(G)$, $|G|$ and $\Delta(G)$, respectively. The degree of a face is the length of its boundary walk. We will write $d(x)$ for $d_G(x)$ when no confusion can arise. A vertex (or face) of degree k is called a *k-vertex* (or *k-face*). We say that two cycles (or faces) are *adjacent* if they share a common edge, respectively. A *triangle* is synonymous with a 3-face. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ in the clockwise order. A cycle C in plane graph G is called a *separating* cycle if both its interior and exterior contain at least one vertex of G. Let $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ represent the sets of vertices located inside and outside C, respectively.

2. Proof of Theorem 3

Suppose that G is a counterexample to Theorem 3 with the least vertices. Without loss of generality, assume that the outside face f_0 is of degree 7, 9, 10, 11 or 12 such that a proper 3-coloring c of the boundary vertices of f_0 cannot be extended to the whole graph G. This implies that there exists at least one vertex in the interior of $b(f_0)$. In fact, $\Delta(G) \geqslant 3$ in this case.

In the sequel, we write C as the boundary walk of f_0, i.e. $C = b(f_0)$. Other faces in G different from f_0 are called *internal faces*. The vertices in C are called *outer vertices* and other vertices *internal vertices*. An internal 3-vertex incident to a 3-face is called *bad*. A face f is called *big* if $d(f) \geqslant 7$.

Parallel to the proof of Theorem 1.2 in [3], we can establish the following Claims 1, 2, 3, 4, 5. For the sake of completeness, we like here to give their proofs.

Claim 1. *G has no separating cycles of length at most 12.*

Proof. Suppose that G has a separating cycle S. Then $|S| \notin \{4, 5, 6, 8\}$ by the assumption. If $|S| \in \{7, 9, 10, 11, 12\}$, we can extend c to $G - \mathrm{int}(S)$ by the minimality of G. Then we delete the (possible) chords from S and extend the 3-coloring of S induced by c to $G - \mathrm{ext}(S)$ by the minimality of G.

Now assume that $|S| = 3$. We claim that both $G - \mathrm{ext}(S)$ and $G - \mathrm{int}(S)$ are 3-colorable, so any 3-coloring of $G - \mathrm{ext}(S)$ can be easily extended to $G - \mathrm{int}(S)$, which contradicts the choice of G. Suppose to the contrary that $G - \mathrm{ext}(S)$ is not 3-colorable. By Theorem 1.1 in [3], $G - \mathrm{ext}(S)$ contains a 7-cycle C_7. Since G does not contain 4, 5, 6 and 8-cycles, C_7 has no chord and thus is 3-colorable. Note that C_7 is not a separating cycle by the above argument, i.e. C_7 is a 7-face. By the minimality of G, we can extend any 3-coloring of C_7 to $G - \mathrm{ext}(S)$. We arrive at a contradiction. By the minimality of G, we can show that $G - \mathrm{int}(S)$ is 3-colorable. This completes the proof of Claim 1. □

Claim 2. *G is 2-connected.*

Proof. The minimality of G asserts that C contains no cut vertex. Assume that B is an end block with a cut vertex $u \in V(G)\backslash V(C)$. We first extend c to $G - (B - u)$, then 3-color B, and finally obtain an extension of c to G. □

Claim 2 implies that there are no vertices of degree less than 2 in G. For each face $f \in F(G)$, $b(f)$ forms a cycle. In particular, C is a cycle of G.

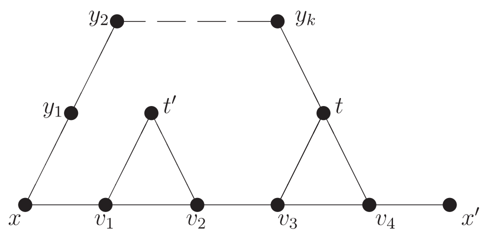

Fig. 1. A tetrad.

Claim 3. *Each 2-vertex in G belongs to C; no 2-vertex in C is incident to a 3-face.*

Proof. If G contains a 2-vertex $v \in V(G)\backslash V(C)$, we first extend c to $G-v$ by the minimality of G, then color v with a color that differs from the colors of its neighbors in G. If a 2-vertex v in C is incident to a 3-face, we first extend c to $G-v$, then recolor v with a color different from those colors of its neighbors in G. We always get a contradiction. □

Claim 4. *No cycle of length 13 or at most 11 in G has a non-triangular chord. In particular, C has no chord at all.*

Proof. If G contains a cycle of length 13 or at most 11 with a non-triangular chord, then it is easy to inspect that G must contain a cycle of length 4, 5, 6 or 8, contradicting the assumption. Thus, the first statement holds.

Suppose that C has a chord e. If e cuts a 3-cycle C_3 from C, then C_3 is a 3-face by Claim 1, which contradicts Claim 3. So suppose that e is a non-triangular chord. It follows that $|C|=12$ and e cuts C into two 7-cycles C' and C''. If both $\mathrm{int}(C')$ and $\mathrm{int}(C'')$ are empty, then it is straightforward to derive that G is 3-colorable. Otherwise, at least one of C' and C'' is a separating 7-cycle, which contradicts Claim 1. Thus, C has no chord. The proof of Claim 4 is complete. □

A *tetrad* is a path $T=v_1v_2v_3v_4$ in the interior of C such that $d(v_i)=3$ for all $i=1,2,3,4$, where $\cdots xTx' \cdots$ is on the boundary of a face and there are triangles $[t'v_1v_2]$ and $[tv_3v_4]$ with $t' \neq x$ and $t \neq x'$.

Claim 5. *G has no tetrad.*

Proof. Suppose to the contrary that G has a tetrad $T=v_1v_2v_3v_4$. Let G' denote the graph obtained from G by deleting vertices v_1, v_2, v_3 and v_4 and identifying x and t. Clearly, G' has neither loops nor multiple edges. Moreover, G' does not contain any face of size of 4, 5, 6 and 8. In order to show that G' belongs to $\mathscr{G}$, we only need to prove that G' does not contain a separating cycle of length 4, 5, 6 or 8. In fact, if $C^*=xy_1y_2\cdots y_{kt}$ is such a separating cycle in G', where $k \in \{3,4,5,7\}$ (see Fig. 1), then $\widetilde{C}=xy_1y_2\cdots y_{kt}v_3v_2v_1x$ is a cycle of length 8, 9, 10 or 12 in G. However, since $t' \notin C^*$ by Claim 4, $\widetilde{C}$ separates t' from v_4 in G, which contradicts Claim 1.

Next, we prove that identifying x and t cannot damage the coloring of C. If this is not true, then it is easy to see that the total distance from x and t to C is at most 1, that is, at least one of x and t lies on C. Without loss of generality, assume that $t \in C$ and let $C=u_1u_2\cdots u_{|C|}u_1$, where the subscripts increase in the clockwise order. Suppose that $u_{|C|}$ is a vertex of C nearest to x. Since $|C| \in \{7,9,10,11,12\}$, C is split by $u_{|C|}$ and t into two paths P_1, P_2 one of which, say $P_1=u_{|C|}u_1\cdots u_jt$, consists of at most six edges. Thus, P_1 and the path $tv_3v_2v_1xu_{|C|}$ yield a cycle D of length at most 11. Since $xv_1v_2v_3v_4x'$is on the boundary of a face, D separates t' from v_4, contradicting Claim 1.

Now, any 3-coloring ϕ of G' can be extended to a 3-coloring of G in this way: first color v_4 and v_3 in succession, then properly color v_1 and v_2. Since x and t have the same color, which implies that x and v_3 have different colors, the required coloring exists. This completes the proof of Claim 5. □

To obtain a contradiction, we rewrite Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$ into the following form:

$$\sum_{v\in V(G)}(d(v)-4)+\sum_{f\in F(G)}(d(f)-4)=-8. \tag{1}$$

Define an initial weight function w by $w(v)=d(v)-4$ for each vertex $v \in V(G)$, $w(f)=d(f)-4$ for each internal face $f \in F(G)\backslash\{f_0\}$ and $w(f_0)=d(f_0)+4$. It is easy to see that $\sum_{x\in V(G)\cup F(G)} w(x)=0$. We shall discharge the initial weight $w(x)$ to its adjacent or incident elements according to the defined rules. During the process, the total sum of weights is fixed. However, after the discharging is finished, the new weight function $w'(x)$ satisfies the following Properties (I) and (II):

(I) $w'(x)\geqslant 0$ for all $x \in V(G)\cup F(G)$;
(II) There exists some $x^* \in V(G)\cup F(G)$ such that $w'(x^*)>0$.

This leads to the following obvious contradiction:

$$0<\sum_{x\in V(G)\cup F(G)} w'(x)=\sum_{x\in V(G)\cup F(G)} w(x)=0. \tag{2}$$

Our discharging rules are defined as follows:

(R1) Every vertex v sends $\frac{1}{3}$ to each incident 3-face.
(R2) Every outer vertex v of degree at least 4 sends $\frac{1}{3}$ to each incident internal big face.
(R3) Every internal vertex v of degree at least 5 sends $\frac{1}{3}$ to each incident big face which is adjacent to two 3-faces incident to v.
(R4) Let f be an internal big face and v a vertex incident to f. We do the following.
(R4a) If v is a 2-vertex or v is a bad 3-vertex, then f sends $\frac{2}{3}$ to v.
(R4b) If v is internal and satisfies one of the following conditions (b1) to (b3), then f sends $\frac{1}{3}$ to v:
(b1) $d(v)=3$ and v is not bad;
(b2) $d(v)=4$ and v is incident to a 3-face which is not adjacent to f;
(b3) $d(v)=4$ and v is incident to two 3-faces each of which is adjacent to f.
(R5) The outside face f_0 sends $\frac{4}{3}$ to each incident vertex.

Let w' denote the resultant weight function after the discharging process is finished according to the rules (R1)–(R5). Since G does not contain cycles of length 4, 5, 6 and 8 and is 2-connected by Claim 2, G contains no 4-face, 5-face, 6-face, 8-face and two adjacent 3-faces.

Let $v \in V(G)$. Then $d(v)\geqslant 2$ by Claim 2.

If $d(v)=2$, then $v \in V(C)$ and v is not incident to any 3-face by Claim 3. Thus, $w'(v)\geqslant 2-4+\frac{4}{3}+\frac{2}{3}=0$ by (R4a) and (R5).

Assume that $d(v)=3$. Then $w(v)=-1$ and v is incident to at most one 3-face. If $v \in V(C)$, then v gets $\frac{4}{3}$ from f_0 by (R5) and sends at most $\frac{1}{3}$ to its incident 3-face by (R1), so that $w'(v)\geqslant -1+\frac{4}{3}-\frac{1}{3}=0$. Now suppose $v \notin V(C)$. If v is bad, then it sends $\frac{1}{3}$ to the unique incident 3-face by (R1) and gets $\frac{2}{3}$ from each of its incident big faces by (R4a). It turns out that $w'(v)\geqslant -1-\frac{1}{3}+2\times\frac{2}{3}=0$. If v is not bad, then v receives $\frac{1}{3}$ from each of incident faces by (b1) in (R4b), and thus $w'(v)\geqslant -1+3\times\frac{1}{3}=0$.

Assume that $d(v)=4$. Then $w(v)=0$ and v is incident to at most two 3-faces. If $v \in V(C)$, then $w'(v)\geqslant \frac{4}{3}-3\times\frac{1}{3}=\frac{1}{3}$ by (R2) and (R5). Suppose that $v \notin V(C)$. If v is not incident to 3-face, then $w'(v)=w(v)=0$. If v is incident to exactly one 3-face, then $w'(v)\geqslant\frac{1}{3}-\frac{1}{3}=0$ by (R1) and (b2) in (R4b). If v is incident to two 3-faces, then $w'(v)\geqslant -2\times\frac{1}{3}+2\times\frac{1}{3}=0$ by (R1) and (b3) in (R4b).

Assuming that $d(v)=5$. Then $w(v)=1$ and v is incident to at most two 3-faces. If $v \in V(C)$, then $w'(v)\geqslant 1+\frac{4}{3}-4\times\frac{1}{3}=1$ by (R1), (R2) and (R5). If $v \notin V(C)$, then $w'(v)\geqslant 1-2\times\frac{1}{3}-\frac{1}{3}=0$ by (R1) and (R3).

Assume that $d(v)\geqslant 6$. Note that v sends at most $\frac{1}{3}$ to each incident face. Thus, $w'(v)\geqslant d(v)-4-\frac{1}{3}d(v)=\frac{2}{3}(d(v)-6)\geqslant 0$.

Let $f \in F(G)$. We see that $d(f)\neq 4, 5, 6, 8$.

If f is the outer face f_0, then $w'(f_0)\geqslant d(f_0)+4-\frac{4}{3}d(f_0)=\frac{1}{3}(12-d(f_0))\geqslant 0$ by (R5) and the fact that $d(f_0)\in\{7, 9, 10, 11, 12\}$.

Suppose that f is an internal face. The proof is divided into the following cases:

(1) If $d(f)=3$, then $w(f)=-1$ and $w'(f)\geqslant -1+3\times\frac{1}{3}=0$ by (R1).

(2) Assume that $d(f)\geqslant 9$. If f is incident to a 2-vertex v, which must belong to C by Claim 3, then f is incident to at least two 3^+-vertices u_1, u_2 of C, i.e. the ends of a maximal path of 2-vertices on the boundary of f. Both u_1 and u_2 receive nothing from f by the rules, and thus $w'(f)\geqslant d(f)-4-\frac{2}{3}(d(f)-2)=\frac{1}{3}(d(f)-8)>0$ by (R4). So suppose that f is not incident to any 2-vertex.
Assume that $d(f)=9$. Then $w(f)=5$. If f sends at most $\frac{1}{3}$ to at least three incident vertices, then $w'(f)\geqslant 5-3\times\frac{1}{3}-6\times\frac{2}{3}=0$ by (R4). If f sends nothing to at least one vertex and $\frac{1}{3}$ to another vertex, then $w'(f)\geqslant 5-\frac{1}{3}-7\times\frac{2}{3}=0$. If f is incident to eight bad 3-vertices, then a tetrad exists in G, which contradicts Claim 5. So, suppose that f is incident to seven bad 3-vertices and the other two vertices x, y each of which is internal and gets $\frac{1}{3}$ from f. To avoid a tetrad in G, x and y must be at distance 4 in the boundary of f. However, in this case, there must exist a vertex of degree at least 4 which receives nothing from f. A contradiction is produced.
Assume that $d(f)=10$. We see that f is incident to at most eight bad 3-vertices, otherwise a tetrad exists in G, contradicting Claim 5. Thus, $w'(f)\geqslant 10-4-8\times\frac{2}{3}-2\times\frac{1}{3}=0$ by (R4).
Assume that $d(f)=11$. It is easy to see that f is incident at most ten bad 3-vertices and therefore $w'(f)\geqslant 11-4-10\times\frac{2}{3}-\frac{1}{3}=0$.
Assume that $d(f)\geqslant 12$. By (R4), $w'(f)\geqslant d(f)-4-\frac{2}{3}d(f)=\frac{1}{3}(d(f)-12)\geqslant 0$.

(3) $d(f)=7$. Then $w(f)=3$, and let $f=[v_1v_2\cdots v_7]$. We first give the following assertion.

Claim 6. *No* 3*-face is adjacent to* f.

Proof. Suppose to the contrary that f is adjacent to a 3-face, say $f'=[v_1uv_2]$. If $u\notin b(f)$, then a 8-cycle $uv_2\cdots v_7v_1u$ is constructed in G, which contradicts the fact that G contains no 8-cycle. So, $u\in b(f)$. Then at least one of edges uv_1 and uv_2 is a chord of $b(f)$, so that $b(f)\cup\{uv_1, uv_2\}$ contains a k-cycle for some $4\leqslant k\leqslant 6$, also a contradiction. This proves Claim 6. □

Claim 6 implies that f is not incident to any bad 3-vertex.

We recall that all 2-vertices belong to $b(f_0)$ by Claim 3. Since $\Delta(G)\geqslant 3$ and C has no chord by Claim 4, f is incident to at most four 2-vertices. Furthermore, by Claim 1 and the fact that G contains no cycles of length 4, 5, 6 or 8, we derive that $b(f)$ contains at most two 2-vertices. Thus, $w'(f)\geqslant 3-2\times\frac{2}{3}-5\times\frac{1}{3}=0$ by (R4a) and (R4b).

The above argument shows Property (I). In order to confirm Property (II), we only need to observe f_0 as well as its adjacent faces or incident vertices.

If $d(f_0)\leqslant 11$, then $w'(f_0)=d(f_0)+4-\frac{4}{3}d(f_0)=\frac{1}{3}(12-d(f_0))>0$ by (R5). So, assume that $d(f_0)=12$. Let v be a vertex incident to f_0. Then v gives at most $\frac{1}{3}$ to each of incident internal faces by (R1) or (R2).

If $d(v)\geqslant 4$, then $w'(v)\geqslant d(v)-4+\frac{4}{3}-\frac{1}{3}(d(v)-1)=\frac{1}{3}(2d(v)-7)>0$ by (R1), (R2) and (R5).

If $d(v)=3$ and v is not incident to 3-face, then $w'(v)=3-4+\frac{4}{3}=\frac{1}{3}$ by (R5).

Thus suppose that every boundary vertex of f_0 is either a 2-vertex or a 3-vertex incident to a 3-face. Let f^* be an internal face adjacent to f_0 with $d(f^*)\geqslant 7$. Since G contains no adjacent 3-faces, such a face f^* exists clearly. Note that $b(f^*)$ contains at least two 3-vertices $x', x''\in b(f_0)$ each is incident to a 3-face by the assumption. It follows that $d(f^*)\geqslant 9$ by Claim 6. As f^* sends nothing to each of x' and x'' by the rules, $w'(f^*)\geqslant d(f^*)-4-\frac{2}{3}(d(f^*)-2)=\frac{1}{3}(d(f^*)-8)>0$.

Up to now, we have shown that there is some $x^*\in V(G)\cup F(G)$ such that $w'(x^*)>0$, i.e. Property (II) holds. □

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